WHAT IS A SERIES III (OF III)

Limit of a Series

Consider the sequence $(a_n)_{n\geq 1}$ with its corresponding series $\sum_{n=1}^{\infty} a_n$. Recall that this series is an infinite sequence $(s_k)_{k\geq 1}$. Hence we can ask about its limit, that is,

$$\lim_{k\to\infty} s_k.$$

It is worthwhile to think about what this limit really means. The element s_k is the sum of the first k elements of the initial sequence $(a_n)_{n\geq 1}$, the element s_{k+1} is the sum of the first $k+1$ elements and so on. In fact we continue adding more and more of the elements of the initial sequence as the index k gets larger. That means, at infinity we get the sum of all elements of the initial sequence $(a_n)_{n\geq 1}$. However it is important to understand that in reality one never reaches infinity. But at least we can take a glimpse at the infinite sum of the sequence elements by studying what happens with the sequence of partial sums s_k, as k gets larger and larger. The notion of limits is the perfect tool to do that.

This idea of interpreting the limit of a series as the sum of all the elements in the initial sequence $(a_n)_{n\geq 1}$ is reflected in the symbol we use for the limit of a series, which is,

$$\sum_{n=1}^{\infty} a_n.$$

We have to be aware that this symbol now has two different meanings which are important to distinguish. On the one hand $\sum_{n=1}^{\infty} a_n$ denotes the series, that is, the sequence

$$s_k = \sum_{n=1}^{k} a_n, \quad \text{for } k \geq 1.$$

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On the other hand, if existent, it denotes the limit of the series, that is

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n.$$

Possible Outcomes for the Limit

There are three possible outcomes for the limit of a series. Firstly the limit is some real number L, in which case we say the series converges to L, and write

$$\sum_{n=1}^{\infty} a_n = L.$$

Secondly the limit is plus or minus infinity, in which case we say the series diverges to plus or minus infinity, respectively, and we write

$$\sum_{n=1}^{\infty} a_n = \infty \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = -\infty.$$

Thirdly the limit does not exist at all, in which case we say the series diverges and we write

$$\sum_{n=1}^{\infty} a_n \text{ does not exist.}$$

The question we have to answer when we deal with series is which one of the three cases occurs for a given initial sequence $(a_n)_{n\geq 1}$.

Example 3.1: The sequence $a_n = 2n$, for $n \geq 1$, corresponds to the series $\sum_{n=1}^{\infty} 2n$, with has the following elements

$$s_1 = 2, \quad s_2 = 6, \quad s_3 = 12, \quad s_4 = 20, \quad s_5 = 30, \quad \ldots$$

Observe how the sequence $(s_k)_{k\geq 1}$ of partial sums diverges to infinity. Indeed for any finite number M there is some k_0 such that $s_k > M$, for all $k \geq k_0$. Hence we can conclude that this series diverges to plus infinity, that is,

$$\sum_{n=1}^{\infty} 2n = \infty.$$