WHAT IS A SERIES II (OF III)

Definition of a Series

Let $(a_n)_{n\geq 1}$ be a sequence and $(s_k)_{k\geq 1}$ the corresponding sequence of partial sums. The sequence of partial sums

$$s_1, \quad s_2, \quad s_3, \quad s_4, \quad \ldots, \quad s_{1000}, \quad \ldots$$

is called the **Series** based on the sequence $(a_n)_{n\geq 1}$. It is common practice to denote the series based on the sequence $(a_n)_{n\geq 1}$ by the symbol

$$\sum_{n=1}^{\infty} a_n.$$

This notation is called *sigma notation* because of the use of the sum sign $\sum$, which is a capital sigma. Observe how this notation tells you everything you need to know about the sequence on which the series is based. It tells you that the underlying sequence is (a_n) whose index n runs from 1 to infinity.

Example 2.1: Consider the sequence $a_n = 2n$, for $n \geq 1$, with the sequence elements

$$a_1 = 2, \quad a_2 = 4, \quad a_3 = 6, \quad a_4 = 8, \quad a_5 = 10, \quad \ldots$$

This sequence $(a_n)_{n\geq 1}$ gives rise to a series $(s_k)_{k\geq 1}$ (that is, the sequence of partial sums) with the following sequence elements

$$s_1 = 2, \quad s_2 = 6, \quad s_3 = 12, \quad s_4 = 20, \quad s_5 = 30, \quad \ldots$$

(Compare with Example 1.2 on the handout What is a Series I.)

Using sigma notation we can express this series as

$$\sum_{n=1}^{\infty} 2n.$$

That means we have a correspondence between the sequence $(2n)_{n\geq 1}$ and the series $\sum_{n=1}^{\infty} 2n$, in the sense that the sequence gives rise to the series and the series is based on the sequence.

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Variations

Recall that the form of the initial sequence may vary. However it is always reflected in the series. For instance if the initial sequence is of the form $(b_m)_{m\geq 0}$, then the partial sums are given by

$$\begin{aligned}
s_0 &= \sum_{m=0}^{0} b_m = b_0 \\
s_1 &= \sum_{m=0}^{1} b_m = b_0 + b_1 \\
s_2 &= \sum_{m=0}^{2} b_m = b_0 + b_1 + b_2 \\
&\vdots \\
s_k &= \sum_{m=0}^{k} b_m, \quad \text{for } k \geq 0.
\end{aligned}$$

Hence the series corresponding to $(b_m)_{m\geq 0}$ is the sequence of partial sums $(s_k)_{k\geq 0}$, and we denoted this series by $\sum_{m=0}^{\infty} b_m$.

Likewise if the initial sequence is of the form $(c_r)_{r\geq 10}$, then the partial sums are given by

$$\begin{aligned}
s_{10} &= \sum_{r=10}^{10} c_r = c_{10} \\
s_{11} &= \sum_{r=10}^{11} c_r = c_{10} + c_{11} \\
s_{12} &= \sum_{r=10}^{12} c_r = c_{10} + c_{11} + c_{12} \\
&\vdots \\
s_k &= \sum_{r=10}^{k} c_r, \quad \text{for } k \geq 10
\end{aligned}$$

Now the series corresponding to $(c_r)_{r\geq 10}$ is the sequence of partial sums $(s_k)_{k\geq 10}$, and we denote this series by $\sum_{r=10}^{\infty} c_r$.