WHAT IS A SERIES I (OF III)

We need a sequence!

A series is always based on a sequence. So let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Then the first five elements in that sequence are

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \ldots$$

It is not relevant that the index of the sequence starts at one. We could equally well consider a sequence $(b_m)_{m\geq 0}$ with the first five elements being

$$b_0 \quad b_1 \quad b_2 \quad b_3 \quad b_4 \quad \ldots$$

or a sequence $(c_r)_{r\geq 10}$ with the first five elements being

$$c_{10} \quad c_{11} \quad c_{12} \quad c_{13} \quad c_{14} \quad \ldots$$

or any other variation. However in the following we will always work with $(a_n)_{n\geq 1}$ as every sequence can be written in this fashion.

Example 1.1: Consider the sequence $a_n = 2n$, for $n \geq 1$. Then the first five elements of this particular sequence are

$$a_1 = 2, \quad a_2 = 4, \quad a_3 = 6, \quad a_4 = 8, \quad a_5 = 10, \quad \ldots$$

Partial Sums

We can use the elements of the sequence $(a_n)_{n\geq 1}$ to create sums. We define them as follows

$$s_k = \sum_{n=1}^{k} a_n, \quad \text{for } k \geq 1.$$

Those sums s_k are called **partial sums** of the initial sequence $(a_n)_{n\geq 1}$. Observe how in each partial sum we add up more and more of the elements in the sequence $(a_n)_{n\geq 1}$. The first partial sum

$$s_1 = \sum_{n=1}^{1} a_n = a_1,$$

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coincides with the first sequence element a_1. The second partial sum

$$s_2 = \sum_{n=1}^{2} a_n = a_1 + a_2,$$

is the sum of the first two elements in the sequence, the third partial sum

$$s_3 = \sum_{n=1}^{3} a_n = a_1 + a_2 + a_3,$$

is the sum of the first three elements, the fourth partial sum

$$s_4 = \sum_{n=1}^{4} a_n = a_1 + a_2 + a_3 + a_4,$$

is the sum of the first four elements and so on. Consequently the k-th partial sum

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k.$$

is the sum of the first k elements of the sequence.

That means we have generated an infinite number of partial sums $s_1, s_2, s_3, \ldots, s_{20}, s_{21}, \ldots, s_{1000}, \ldots$. Note that these partial sums form a sequence, which is called the **sequence of the partial sums**.

Also observe that each partial sum can be expressed in terms of its predecessor, as clearly

$$s_{k+1} = s_k + a_{k+1}, \quad \text{for } k \geq 1.$$

Example 1.2: We continue with Example 1.1. There we get the following partial sums

$$\begin{aligned}
s_1 &= 2 \\
s_2 &= 2 + 4 = 6 \\
s_3 &= 2 + 4 + 6 = 12 \\
s_4 &= 2 + 4 + 6 + 8 = 20 \\
s_5 &= 2 + 4 + 6 + 8 + 10 = 30. \\
&\vdots
\end{aligned}$$

and thus the sequence of the partial sums is

$$2, \quad 6, \quad 12, \quad 20, \quad 30, \quad \ldots$$